On complete affine structures in Lie groups *

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Abstract

Left invariant affine structures in a Lie group G are in one-to-one correspondence with left-symmetric algebras over its Lie algebra $\mathfrak{g} = T_eG$ ("over" means that the commutator $[x, y] = xy - yx$ coincides with the Lie brackets; left-symmetric algebras can be defined as Lie-admissible algebras such that the multiplication by left defines a representation of the underlying Lie algebra). An affine structure (and the corresponding left symmetric algebra) is complete if G is affinely equivalent to $\mathfrak{g}$. By the main result of this paper, a complete left symmetric algebra admits a canonical decomposition: there is a Cartan subalgebra $\mathfrak{h}$ such that the root subspaces for the representations L (by left multiplications) and ad coincide. Then operators $L(x)$ and $\mathrm{ad}(x)$ have equal semisimple parts for all $x \in \mathfrak{h}$. This decomposition is unique. For simple complete left-symmetric algebras whose canonical decomposition consists of one dimensional spaces we define two types of graphs and prove some their properties. This makes possible to describe, for dimensions less or equal to 5, these graphs and algebras.

A connection in a manifold G can be defined by the covariant derivative $\nabla_\xi \eta$, where ξ, η, and $\nabla_\xi \eta$, are smooth vector fields. If the torsion and the curvature vanish

$$\nabla_\xi \eta - \nabla_\eta \xi - [\xi, \eta] = 0,$$
$$\nabla_\xi \nabla_\eta \zeta - \nabla_\eta \nabla_\xi \zeta - \nabla_{[\xi,\eta]} \zeta = 0,$$

where $[\xi, \eta]$ is the Lie bracket of vector fields, then the manifold admits an atlas of coordinate charts with affine coordinate transformations. This defines the *affine structure* in G. If G is a Lie group and ∇, ξ, η are left invariant then the field $\nabla_\xi \eta$ is also left invariant. Realizing the Lie algebra $\mathfrak{g}$ by these vector fields we get an operation in $\mathfrak{g}$ which satisfies identities

$$\xi\eta - \eta\xi = [\xi, \eta], \tag{1}$$
$$[\xi, \eta]\zeta = \xi(\eta\zeta) - \eta(\xi\zeta). \tag{2}$$

*This is a translation of the paper published in book: Essays in differential equations and algebra, IITPM, Omsk (1992), 67-80 (in Russian). There are nonessential changes in the exposition, references are extended but the mathematical content is identical. In the case of the field $\mathbb{C}$, Section 1 overlaps with the paper [2].

The second can be written as follows:

$$(\xi, \eta, \zeta) = (\eta, \xi, \zeta), \tag{3}$$

where $(\xi, \eta, \zeta) = \xi(\eta\zeta) - (\xi\eta)\zeta$ is the *associator*. Algebras satisfying (3) are called *left-symmetric*. They were defined by Vinberg[1] in the paper [4] which contains the classification of homogeneous convex pointed cones. Left-symmetric algebras are *Lie–admissible*: (3) implies that the brackets in (1) satisfies the Jacobi identity. By (2),

$$L([\xi, \eta]) = [L(\xi), L(\eta)], \tag{4}$$

where $L(\xi)\zeta = \xi\zeta$. Thus left symmetric algebras could be defined as Lie–admissible algebras such that the left multiplication is a representation of the underlying Lie algebra. There is a natural realization of a left-symmetric algebra $\mathfrak{g}$ by affine vector fields on the linear space $\mathfrak{g}$: each $x \in \mathfrak{g}$ corresponds the field

$$F_x(y) = xy + x. \tag{5}$$

For the multiplication defined by the linear part of (5), identities (1) and (2) are equivalent to the condition that the set

$$\{F_x\colon\ x \in \mathfrak{g}\} \tag{6}$$

is a Lie algebra with respect to the Lie bracket of vector fields. Let G be the simply connected Lie group corresponding to $\mathfrak{g}$. The realization above defines an action of G by affine transformations of $\mathfrak{g}$. Since the mapping $x \to F_x(o)$ is nondegenerate, the stable subgroup of the origin o is discrete. This defines a natural affine structure in G. If the action is transitive on $\mathfrak{g}$ then it is free since the stable subgroup is trivial being isomorphic to the fundamental group of the orbit. Then G is diffeomorphic to Euclidean space. Therefore, G is solvable in this case (note that G cannot be isomorphic to the universal covering group of $SL(2, \mathbb{R})$ being linear). We say that the affine structure and the left-symmetric algebra are *complete* if the action above is transitive. This is equivalent to the geodesical completeness.

The paper is organized as follows. In Section 1, we give the algebraic criterion of the completeness for real left-symmetric algebras (for complex ones it is known, see [2]). Section 2 contains the main result of the paper. We prove that a complete left-symmetric algebra admits a *canonical decomposition*: there exists a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that the root decompositions for the representations L and ad coincide as well as semisimple parts of operators $L(x)$ and $\mathrm{ad}(x)$ for all $x \in \mathfrak{h}$. The decomposition is unique. In the third section, we consider simple left-symmetric algebras that admit one dimensional root decompositions. We define two types of graphs for them and prove some their properties. This makes possible to describe the graphs and give the complete list of these algebras in dimensions less than 6 [2].

[1] and by many other authors in various forms, see a recent survey [6].

[2] a computation shows that in dimension 6 there is a lot of graphs with these properties as well as simple algebras of this type.

All algebras are finite dimensional, real or complex. We denote the field by $\mathbb{F}$. In most cases, the field is not specified since arguments hold for both cases. By $\mathfrak{g}$ we denote the left-symmetric algebra as well as the underlying Lie algebra, by G the corresponding Lie group of affine transformations of $\mathfrak{g}$.

There is one more equivalent version of (3):

$$[L(\xi), R(\zeta)] = R(\xi\zeta) - R(\zeta)R(\xi), \tag{7}$$

where $R(\xi)\eta = \eta\xi$ is the operator of the right multiplication by ξ.

1 Algebraic criterion of completeness

Any left-symmetric algebra $\mathfrak{g}$ can be extended by the addition of the identity element $\mathbf{1}$ which satisfies $\mathbf{1} \cdot x = x \cdot \mathbf{1} = x$ for all $x \in \mathfrak{g}$. A simple straightforward calculation shows that the extended algebra $\mathfrak{g}_{\mathbf{1}} = \mathbb{F} \cdot \mathbf{1} + \mathfrak{g}$ satisfies (3). In geometric terms, this means that the affine action in the vector space $\mathfrak{g}$ naturally defines a linear representation of G in the vector space $\mathfrak{g}_{\mathbf{1}}$: any affine transformation $x \to Ax + b$ of $\mathfrak{g}$ in the hyperplane $\mathbf{1} + \mathfrak{g}$ of vectors $\mathbf{1} + x$, $x \in \mathfrak{g}$, uniquely extends to a linear transformation of $\mathfrak{g}_{\mathbf{1}}$. Vector fields (5) on $\mathbf{1} + \mathfrak{g}$ have the form $F_x(y) = x(\mathbf{1} + y)$ and are included to the family

$$\{F_x\colon\ x \in \mathfrak{g}_{\mathbf{1}}\}, \tag{8}$$

where $F_x(y) = xy$, $y \in \mathfrak{g}_{\mathbf{1}}$. On the level of Lie algebras, $\mathfrak{g}_{\mathbf{1}}$ is the trivial one dimensional extension of $\mathfrak{g}$. Let us equip with the index $\mathbf{1}$ spaces and groups relating to $\mathfrak{g}_{\mathbf{1}}$; for instance, $G_{\mathbf{1}}$ is the linear group in the linear space $\mathfrak{g}_{\mathbf{1}}$ corresponding to (8). Set

$$\begin{aligned} P_{\mathbf{1}}(x) &= \det(R_{\mathbf{1}}(x)), \\ P(x) &= \det(I + R(x)), \end{aligned} \tag{9}$$

where I is the identical transformation. The polynomial P is the restriction of $P_{\mathbf{1}}$ to the hyperplane $\mathbf{1} + \mathfrak{g}$.

Lemma 1. *For all $x, y \in \mathfrak{g}_{\mathbf{1}}$*

$$R(\mathrm{e}^{L(y)}x) = \mathrm{e}^{L(y)}R(x)\mathrm{e}^{-\operatorname{ad}(y)}. \tag{10}$$

Proof. Differentiating (10) on y at $y = 0$ we get

$$R(L(u)x) = L(u)R(x) - R(x)\operatorname{ad}(u), \quad u \in \mathfrak{g}_{\mathbf{1}}. \tag{11}$$

This is equivalent to the equality

$$z(ux) = u(zx) - [u, z]x$$

for all $u, x, z \in \mathfrak{g}_{\mathbf{1}}$ that is the same as (4). Since left and right sides of (10) define representations of $G_{\mathbf{1}}$ and the equality holds for $y = 0$, (10) follows from (11). $\square$

Corollary 1. *For all $x, y \in \mathfrak{g}_{\mathbf{1}}$*

$$P_{\mathbf{1}}(\mathrm{e}^{L(y)}x) = \mathrm{e}^{\mathrm{Tr}(R(y))}P_{\mathbf{1}}(x), \tag{12}$$

Proof. Since $\mathrm{ad}(y) = L(y) - R(y)$, the right side of (12) is the determinant of the right side in (10). $\square$

According to (12), the polynomial $P_{\mathbf{1}}$ is an eigenfunction of $G_{\mathbf{1}}$:

$$P_{\mathbf{1}}(g(x)) = \lambda(g)P_{\mathbf{1}}(x), \tag{13}$$

where λ is some one dimensional character of $G_{\mathbf{1}}$. For $x = \mathbf{1}$, $g = \exp(y)$, where $\exp : \mathfrak{g}_{\mathbf{1}} \to G_{\mathbf{1}}$ is the exponential mapping, setting $\mathrm{e}^y = \exp(y)(\mathbf{1})$ we get

$$\lambda(\exp(y)) = P_{\mathbf{1}}(\mathrm{e}^y) = \mathrm{e}^{\mathrm{Tr}(R(y))}$$

(note that $P_{\mathbf{1}}(\mathbf{1}) = 1$).

Lemma 2. *If $\mathfrak{g}$ is complete then the operator $L(y)$ is degenerate for all $y \in \mathfrak{g}$.*

Proof. Otherwise, the operator $\mathrm{e}^{L(y)} - I$ is nondegenerate for generic $y \in \mathfrak{g}$. Then any affine mapping with the linear part $\mathrm{e}^{L(y)}$ has a fixed point contradictory to the assumption that $\mathfrak{g}$ is complete since transitive actions are free. $\square$

Proposition 1. *The left-symmetric algebra $\mathfrak{g}$ is complete if and only if any of the following conditions holds for all $x \in \mathfrak{g}$:*

(a) *$R(x)$ is nilpotent;*

(b) *$P(x) = 1$;*

(c) *$P(x) \neq 0$;*

(d) *$\mathrm{Tr}\, R(x) = 0$.*

Proof. Implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c), (a) $\Rightarrow$ (d) are evident. It follows from (12) that (b) and (d) are equivalent. If $\det(I + tR(x)) = 1$ for all $t \in \mathbb{F}$ then $R(x)$ is nilpotent; thus (a) follows from (b). Hence it is sufficient to prove that (c) implies the completeness of $\mathfrak{g}$ and that (d) is true if $\mathfrak{g}$ is complete.

Since vectors of vector fields (6) at y (5) have the form $xy + x = (I + R(y))x$, the condition (c) is equivalent to the assumption that the tangent space to the orbit of y is equal to $\mathfrak{g}$. Therefore, all orbits are open. Then they are closed; hence there is only one orbit.

Let $\mathfrak{g}$ be complete, $y \in \mathfrak{g}$, $\mathfrak{g}^0$ be the root subspace of the eigenvalue 0 of $L(y)$ and $\mathfrak{g}^1$ be the sum of all other root subspaces. Then $\mathfrak{g}_{\mathbf{1}} = \mathfrak{g}^0 \oplus \mathfrak{g}^1$. Since $\mathfrak{g} \cdot \mathfrak{g}_{\mathbf{1}} \subseteq \mathfrak{g}$, the inclusion $\mathfrak{g}^1 \subseteq \mathfrak{g}$ holds. Therefore,

$$\mathfrak{g}^0 \cap (\mathbf{1} + \mathfrak{g}) \neq \varnothing.$$

Hence there exist $x \in \mathbf{1} + \mathfrak{g}$ and $n \in \mathbb{N}$ such that $L^n(y)x = 0$. By (12),

$$P_{\mathbf{1}}(\mathrm{e}^{tL(y)}x) = \mathrm{e}^{t\,\mathrm{Tr}\,R(y)}P_{\mathbf{1}}(x) \tag{14}$$

for all $t \in \mathbb{F}$. The left side of (14) is a polynomial but the right one is an exponent on t. Hence either $P_{\mathbf{1}}(x) = 0$ or $\mathrm{e}^{t\,\mathrm{Tr}\,R(y)} = \mathrm{const}$. The assumption $P_{\mathbf{1}}(x) = 0$ contradicts to the completeness of $\mathfrak{g}$ and (13). This proves the proposition. $\square$

Remark 1. If $\mathbb{F} = \mathbb{C}$ and $\mathfrak{g}$ is complete then (d) is an evident consequence of (c) since any nonconstant polynomial has zeroes.

Corollaries below follow from Proposition 1, (a) and (d), respectively.

Corollary 2. *Subalgebras and quotient algebras of a complete left-symmetric algebra are complete.* $\square$

Corollary 3. *A real left-symmetric algebra $\mathfrak{g}$ is complete if and only if $\mathfrak{g} \otimes \mathbb{C}$ is complete.* $\square$

2 The canonical decomposition

Let $\mathfrak{h}$ be a Cartan subalgebra of the Lie algebra $\mathfrak{g}$. If $\mathfrak{g}$ is complex and λ is a linear functional on $\mathfrak{h}$ then $\mathfrak{h}^\lambda$, $\mathfrak{g}^\lambda$ denote root subspaces of λ for representations ad, L, respectively. For real $\mathfrak{g}$, λ is a linear functional on $\mathfrak{h}^{\mathbb{C}} = \mathfrak{h} + i\mathfrak{h}$ and $\mathfrak{h}^\lambda$, $\mathfrak{g}^\lambda$ are real parts of the corresponding root spaces. Then

$$\mathfrak{g} = \sum_\lambda \oplus\, \mathfrak{h}^\lambda, \tag{15}$$

$$\mathfrak{g} = \sum_\lambda \oplus\, \mathfrak{g}^\lambda. \tag{16}$$

We say that the decomposition is *canonical* if (15) coincides with (16) (i. e. $\mathfrak{h}^\lambda = \mathfrak{g}^\lambda$ for all λ). The space $\mathfrak{h}_{\mathbf{1}} = \mathbb{F}\cdot\mathbf{1} + \mathfrak{h}$ is the Cartan subalgebra for $\mathfrak{g}_{\mathbf{1}}$ (and the trivial one dimensional extension of $\mathfrak{h}$). Similar decompositions hold for $\mathfrak{h}_{\mathbf{1}}$ and $\mathfrak{g}_{\mathbf{1}}$:

$$\mathfrak{g}_{\mathbf{1}} = \sum_\lambda \oplus\, \mathfrak{h}_{\mathbf{1}}^\lambda, \tag{17}$$

$$\mathfrak{g}_{\mathbf{1}} = \sum_\lambda \oplus\, \mathfrak{g}_{\mathbf{1}}^\lambda \tag{18}$$

Clearly,

$$\mathfrak{h}_{\mathbf{1}}^\lambda = \mathfrak{h}^\lambda, \quad \mathfrak{g}_{\mathbf{1}}^\lambda = \mathfrak{g}^\lambda, \quad \lambda \neq 0; \qquad \mathfrak{h}_{\mathbf{1}}^0 = \mathfrak{h}_{\mathbf{1}}. \tag{19}$$

In general, $\mathbf{1} \notin \mathfrak{g}_{\mathbf{1}}^0$.

Lemma 3. *The decomposition (16) is canonical if and only if $\mathbf{1} \in \mathfrak{g}_{\mathbf{1}}^0$.*

Proof. If (16) is canonical then $\mathfrak{h}^0 = \mathfrak{g}^0$. Since $\mathfrak{h}\cdot\mathbf{1} = \mathfrak{h} = \mathfrak{h}^0 = \mathfrak{g}^0$, we get $\mathbf{1} \in \mathfrak{g}_{\mathbf{1}}^0$.

It is sufficient to prove the converse for $\mathbb{F} = \mathbb{C}$ (the case $\mathbb{F} = \mathbb{R}$ can be easily reduced to it). Since L is a representation, for all linear functionals λ, μ

$$\mathfrak{h}_{\mathbf{1}}^{\lambda} \cdot \mathfrak{g}_{\mathbf{1}}^{\mu} \subseteq \mathfrak{g}_{\mathbf{1}}^{\lambda+\mu}. \tag{20}$$

If $\mathbf{1} \in \mathfrak{g}_{\mathbf{1}}^{0}$ then $\mathfrak{h}_{\mathbf{1}}^{\lambda} \subseteq \mathfrak{h}_{\mathbf{1}}^{\lambda} \cdot \mathfrak{g}_{\mathbf{1}}^{0}$. By (20), $\mathfrak{h}_{\mathbf{1}}^{\lambda} \subseteq \mathfrak{g}_{\mathbf{1}}^{\lambda}$. Therefore, each summand in (17) is contained in the corresponding summand of (18); since the sums are equal, the summands coincide. For (15) and (16), the assertion holds due to (19). $\square$

Lemma 4. *Let $g \in G_{\mathbf{1}}$. Then*

$$\mathfrak{g} = \sum_{\lambda} \oplus\, g(\mathfrak{g}_{\mathbf{1}}^{\lambda})$$

is the decomposition (18) for the Cartan subalgebra $\mathrm{Ad}(g)\mathfrak{h}$.

Proof. This is true since $L(\mathrm{Ad}(g)h) = gL(h)g^{-1}$. $\square$

Lemma 5. *If the decomposition (16) is canonical then*

$$\mathfrak{g}^{\lambda} \cdot \mathfrak{g}^{\mu} \subseteq \mathfrak{g}^{\lambda+\mu} \tag{21}$$

for all λ, μ. In particular, $\mathfrak{g}^{0}$ is a subalgebra of the left-symmetric algebra $\mathfrak{g}$.

Proof. The inclusion holds due to (19), (20) and the definition of the canonical decomposition. $\square$

Theorem 1. *Each complete left-symmetric algebra admits the unique canonical decomposition.*

Proof. Let $\mathfrak{h}$ be any Cartan subalgebra. By (19), $\mathfrak{g}_{\mathbf{1}}^{\lambda} \subseteq \mathfrak{g}$ if $\lambda \neq 0$ in (18). Hence

$$\mathfrak{g}_{\mathbf{1}}^{0} \cap (\mathbf{1} + \mathfrak{g}) \neq \varnothing.$$

If $x \in \mathfrak{g}_{\mathbf{1}}^{0} \cap (\mathbf{1} + \mathfrak{g})$ then $g(x) = \mathbf{1}$ for some $g \in G$ (by the definition of the completeness; recall that the affine action of G may be realized in the hyperplane $\mathbf{1} + \mathfrak{g}$). It follows from Lemma 4 that for the decomposition (16) with respect to the Cartan subalgebra $\mathrm{Ad}(g)\mathfrak{h}$ the condition $\mathbf{1} \in \mathfrak{g}_{\mathbf{1}}^{0}$ holds. This makes possible to apply Lemma 3.

To prove the uniqueness, we use the following known fact: any two Cartan subalgebras in a (real) solvable Lie algebra are conjugated by an inner automorphism (if the left symmetric algebra $\mathfrak{g}$ is complete then the Lie algebra $\mathfrak{g}$ is solvable; this is proved, for example, in [2]; see also the introduction). Suppose that $\mathfrak{h}$ and $\tilde{\mathfrak{h}}$ are Cartan subalgebras corresponding to canonical decompositions (16). We claim that $\mathfrak{h} = \tilde{\mathfrak{h}}$. There exists $g \in G$ such that $\tilde{\mathfrak{h}} = \mathrm{Ad}(g)\mathfrak{h}$. Then $\tilde{\mathfrak{g}}^{0} = g(\mathfrak{g}^{0})$ by Lemma 4 and $\mathbf{1} \in \tilde{\mathfrak{g}}^{0} \cap \mathfrak{g}^{0}$ by Lemma 3. Hence $\mathbf{1}, g(\mathbf{1}) \in \tilde{\mathfrak{g}}^{0}$. Since the action of G is free, the condition $\mathbf{1} \to g(\mathbf{1})$ uniquely determines $g \in G$. By Lemma 5 and Corollary 2, $\tilde{\mathfrak{g}}^{0}$ is a complete left symmetric algebra. Hence the subgroup $\tilde{G}^{0}$, corresponding to $\tilde{\mathfrak{g}}^{0}$ contains the unique transformation that sends $\mathbf{1}$ to $g(\mathbf{1})$. Therefore, $g \in \tilde{G}^{0}$. Thus

$$\mathfrak{h} = \mathrm{Ad}(g^{-1})\tilde{\mathfrak{h}} = \mathrm{Ad}(g^{-1})\tilde{\mathfrak{g}}^{0} = \tilde{\mathfrak{g}}^{0} = \tilde{\mathfrak{h}}. \qquad \square$$

Remark 2. Due to the uniqueness, each automorphism of a complete left symmetric algebra keeps the canonical decomposition. On the other hand, the existence of the canonical decomposition for a complete left symmetric algebra which is not nilpotent as a Lie algebra implies the existence of a non-discrete group of automorphisms. Indeed, the semisimple parts of the representations L and ad (they coincide) are differentiations of $\mathfrak{g}$ according to (21).

3 Graphs of one dimensional canonical decompositions

In this section, we assume that $\mathbb{F} = \mathbb{C}$ and

$$\dim \mathfrak{g}^{\lambda} \leq 1 \tag{22}$$

for all spaces $\mathfrak{g}^{\lambda}$ in the decomposition (16). Let $e_0 \in \mathfrak{g}^0$, $e_0 \neq 0$. The spectrum of the linear transformation $L(e_0)$ may be identified with the set of all nontrivial roots; let us denote it by Λ. For $\lambda \in \Lambda$ let e_λ be the eigenvector (for $\lambda = 0$, the chosen vector e_0 satisfies this). In the base $\mathcal{B} = \{e_\lambda \colon \lambda \in \Lambda\}$ the multiplication is defined by relations

$$e_\lambda e_\mu = c_{\lambda,\mu} e_{\lambda+\mu}. \tag{23}$$

As usual, we assume that $c_{\lambda,\mu} = c_{\mu,\lambda} = 0$ and $e_\lambda = 0$ if $\lambda \notin \Lambda$.

Each one dimensional canonical decomposition corresponds to graphs Γ_l and Γ_r relating to left and right multiplications. The set of vertices for both graphs is Λ; it is natural to assume that 0 is a base point. The graph Γ_l contains the edge from λ to μ if and only if

$$c_{\mu-\lambda,\lambda} \neq 0. \tag{24}$$

For Γ_r, (24) is replaced by

$$c_{\lambda,\mu-\lambda} \neq 0. \tag{25}$$

By definition, $e_0 e_\lambda = \lambda e_\lambda$. Hence $c_{0,\lambda} = \lambda$. In particular,

$$c_{0,\lambda} \neq 0 \quad \text{if} \quad \lambda \neq 0. \tag{26}$$

Therefore,

`(l1)` there is the loop at each nonzero vertex in Γ_l;

`(r1)` the vertex 0 in Γ_r is joined with all other vertices.

Graphs Γ_l and Γ_r mutually determine each other by the following procedure. Let Γ_l (Γ_r) contains the edge $\overline{\lambda\mu}$. Let us remove λ to 0 and join the endpoint of the resulting vector with the endpoint of the original one. Doing this with all edges in Γ_l (Γ_r) we get Γ_r (Γ_l).

It follows from (24), (25) that

`(l2)` if Γ_l contains the edge $\overline{\lambda\mu}$ then it contains the vertex $\mu-\lambda$,

and the same property `(r2)` of Γ_r. Let us abbreviate the notation

$$L_\lambda = L(e_\lambda), \quad R_\lambda = R(e_\lambda).$$

All operators R_λ are nilpotent by Proposition 1. Hence the assumption (22) implies

$$R_0 = 0. \tag{27}$$

Therefore,

`(l3)` there are no edge that comes out of the vertex 0 in Γ_l;

`(r3)` Γ_r contains no loops.

As a consequence of (26) and (27) we get

$$[\mathfrak{g},\mathfrak{g}] = \sum_{\lambda\neq 0} \oplus\, \mathfrak{g}^\lambda.$$

Since the Lie algebra $\mathfrak{g}$ is solvable, there exists a base in the linear space $\mathfrak{g}$ such that $L([\mathfrak{g},\mathfrak{g}])$ is strictly triangular in it; in particular, this is true for all L_λ, where $\lambda \neq 0$. Hence the product $L_{\lambda_1}L_{\lambda_2}\dots L_{\lambda_m}$ is nilpotent if $\lambda_k \neq 0$ for some $k\in\{1,\dots,m\}$. On the other hand, if $\lambda_1+\dots+\lambda_m = 0$ then $L_{\lambda_1}L_{\lambda_2}\dots L_{\lambda_m}$ is diagonal in the base $\mathcal{B}$. Therefore, if there is a nonzero summand in this sum then

$$L_{\lambda_1}\dots L_{\lambda_m} = 0 \tag{28}$$

In particular, $L_\lambda L_{-\lambda} = L_{-\lambda}L_\lambda = 0$ for all $\lambda\neq 0$; hence

$$[L_\lambda, L_{-\lambda}] = L([e_\lambda, e_{-\lambda}]) = 0.$$

Since $[e_\lambda, e_{-\lambda}] = (c_{\lambda,-\lambda} - c_{-\lambda,\lambda})e_0$,, by (26),

$$[e_\lambda, e_{-\lambda}] = 0. \tag{29}$$

Thus $c_{\lambda,-\lambda} = c_{-\lambda,\lambda}$. This means that

`(l4)` if some edge in Γ_l comes to 0 from λ then the same is true for the vertex $-\lambda$ (in particular, Γ_l contains this vertex).

Clearly, the analogous property `(r4)` holds for Γ_r. Furthermore, (28) immediately implies that

`(l5)` Γ_l has no cycle.

Suppose that Γ_l contains two consecutive edges, say, $\overline{\lambda\mu}$ and $\overline{\mu\nu}$. Then

$$e_{\nu-\mu}(e_{\mu-\lambda}e_\lambda) \neq 0.$$

By (2),

$$e_{\nu-\mu}(e_{\mu-\lambda}e_\lambda) = e_{\mu-\lambda}(e_{\nu-\mu}e_\lambda) + [e_{\nu-\mu}, e_{\mu-\lambda}]e_\lambda.$$

Since at least one summand in the right side above is nontrivial, we get

`(l6)` each pair of consecutive edges in Γ_l can be included either to a triangle or to a parallelogram in Γ_l.

The parallelogram may be degenerate but new edges coincides with initial ones only if $\mu - \lambda = \nu - \mu$.

It follows from (7) and Proposition 1 that

$$\mathrm{Tr}(R(x)R(y)) = 0 \tag{30}$$

for all $x, y \in \mathfrak{g}$. Analogously, the equality

$$[L(y), R(x)^2] = [L(y), R(x)]R(x) + R(x)[L(y), R(x)],$$

taken together with (7) and (30) implies that

$$\mathrm{Tr}(R(x)^2 R(y)) = 0.$$

In particular, $\mathrm{Tr}(R_\lambda^2 R_{-2\lambda}) = 0$ for all $\lambda \in \Lambda$. Therefore,

`(r5)` Γ_r cannot contain exactly one path of the type

$$\mu \to \mu - 2\lambda \to \mu - \lambda \to \mu.$$

We formulate also several properties that hold only for simple left symmetric algebras. A subspace $\mathfrak{j} \subseteq \mathfrak{g}$ is an *ideal* in the left symmetric algebra $\mathfrak{g}$ if $\mathfrak{j}\mathfrak{g} \subseteq \mathfrak{j}$ and $\mathfrak{g}\mathfrak{j} \subseteq \mathfrak{j}$ (i. e. if $\mathfrak{j}$ is a two-side ideal); an algebra is *simple* if it contains no proper ideals.

`(s1)` For any $\lambda \in \Lambda$, the graph Γ_l contains an edge parallel to $\overline{0\lambda}$; the graph Γ_r has an edge that comes out of λ.

These properties are mutually dual with respect to the procedure described above. They hold since the kernel of the representation L is an ideal in $\mathfrak{g}$.

`(s2)` The vertex 0 is reachable from any point $\lambda \in \Lambda$ in the union of graphs Γ_l and Γ_r.

The set $\mathcal{R}$ of all vertices that are reachable from λ defines a nontrivial ideal which must coincide with $\mathfrak{g}$; hence $\mathcal{R}$ contains e_0.

`(s3)` Γ_l and Γ_r contain at least one pair of symmetric vertices $\lambda, -\lambda$ and edges from them to 0.

The assertion follows from `(s2)`, `(l4)` and `(r4)`.

The properties above distinguish graphs of complex simple algebras for dimensions ≤ 5. Clearly, graphs which can be identified by rotations and dilations are equivalent (these operations corresponds to the multiplication of e_0 by a complex number). Thus we may assume that 1 is a vertex and that Γ_l contains no vertex $\lambda \neq 0$ such that $|\lambda| < 1$. Also, we drop trivial edges defined in `(l1)`.

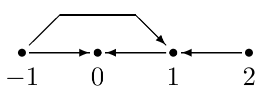
There is no simple algebras of dimension 2.

In dimension 3, there is exactly one graph that is determined by `(s3)`. The corresponding algebra is unique up to an isomorphism and can be defined by relations

$$e_0e_1 = e_1, \quad e_0e_{-1} = -e_{-1}, \quad e_1e_{-1} = e_{-1}e_1 = e_0$$

(Auslander's algebra; it defines an affine action that gives an example of a free transitive group of affine motions that contains no translation, see [3]).

In dimension 4, there is also only one graph Γ_l



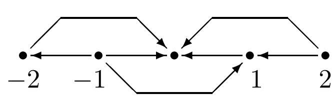
and the unique up to an isomorphism simple algebra

$$e_0e_{-1} = -e_{-1}, \quad e_0e_1 = e_1, \quad e_0e_2 = 2e_2,$$
$$e_{-1}e_1 = e_1e_{-1} = e_0, \quad e_2e_{-1} = 2e_1, \quad e_{-1}e_2 = e_1.$$

In dimension 5, there is one parameter family of graphs and simple algebras. Graphs consists of two pairs of symmetric vertices and edges coming to 0. Algebras can be defined by relations

$$e_0e_{-1} = -e_{-1}, \quad e_0e_1 = e_1, \quad e_{-1}e_1 = e_1e_{-1} = e_0,$$
$$e_0e_{-\lambda} = -\lambda e_{-\lambda}, \quad e_0e_\lambda = \lambda e_\lambda, \quad e_{-\lambda}e_\lambda = e_\lambda e_{-\lambda} = e_0.$$

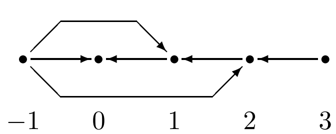
For $\lambda = 2$, some additional relations may hold:



$$e_2e_{-1} = \alpha e_1,$$
$$e_{-1}e_2 = \beta e_1,$$
$$e_{-1}e_{-1} = \gamma e_{-2},$$

where $2\alpha = \beta + \gamma$. For $\alpha = 0$, $\beta = 0$, $\gamma = 0$ edges $-1 \to 1$, $2 \to 1$, $-1 \to -2$ must be dropped, respectively. Algebras corresponding to triples (α, β, γ) $(\alpha', \beta', \gamma')$ are isomorphic if and only if vectors (α, β, γ) and $(\alpha', \beta', \gamma')$ are collinear (see Remark 2). Thus the family corresponding to $\lambda = 2$ is modelled by the projective line $\mathbb{FP}^1$ with three distinguished points.

There is only one simple algebra with one dimensional root decomposition which is not mentioned above:



$$e_0e_k = ke_k, \quad e_{-1}e_k = ke_{k-1},$$
$$e_ke_{-1} = e_{k-1}, \quad k = 1, 2, 3;$$
$$e_0e_{-1} = -e_{-1}.$$

It is included to infinite series and an infinite dimensional left symmetric algebra with the same relations and k running over $\mathbb{N}$.

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